GLOBAL SMALL SOLUTIONS TO A TROPICAL CLIMATE MODEL WITHOUT THERMAL DIFFUSION

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Abstract. We obtain the global well-posedness of classical solutions to a tropical climate model derived by Feireisl-Majda-Pauluis in [7] with only the dissipation of the first baroclinic model of the velocity ($-\eta\Delta v$) under small initial data. The main difficulty is the absence of thermal diffusion. To overcome it, we exploit the structure of the equations coming from the coupled terms, dissipation term and damp term. Then we find the hidden thermal diffusion. In addition, based on the Littlewood-Palay theory, we establish a generalized commutator estimate, which may be applied to other partial differential equations.

1. Introduction

The purpose of this article is to study the cauchy problem for a tropical model without thermal diffusion:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \alpha u + \nabla p = -\mathrm{div}(v\otimes v), \\ \partial_t v + u\cdot\nabla v + v\cdot\nabla u + \alpha v - \eta\Delta v = \nabla\theta, \\ \partial_t\theta + u\cdot\nabla\theta = \mathrm{div}v, \\ \mathrm{div}u = 0, \\ (u(0,x), v(0,x), \theta(0,x)) = (u_0(x), v_0(x), \theta_0(x)), \end{cases} \tag{1.1}$$

here $(t,x)\in\mathbb{R}^+\times\mathbb{R}^2$, $u=(u^1,u^2)$, $v=(v^1,v^2)$ stand for the barotropic mode and the first baroclinic mode of the vector velocity, respectively, p,θ represent the scalar pressure, scalar temperature, respectively, α and η are the nonnegative parameters.

By performing a Galerkin truncation to the hydrostatic Boussinesq equations, Feireisl-Majda-Pauluis in [7] derived a version of (1.1) without any Laplacian terms, of which the first baroclinic mode had been originally used in some studies of tropical atmosphere in [8] and [17]. For more details on the first baroclinic mode, we refer to the section 1 and section 2 in [7] and references therein.

Recently, for the version of (1.1) with $-\Delta u$, $\alpha=0$ and $\eta=1$, Li-Titi in [15] obtained the global well-posedness without any small assumptions of initial data. The difficulty of their work is that energy method can not be applied to get the gradient estimate of (u,v,θ) directly due to the absence of thermal diffusion. However, by introducing a unknown w,

$$w :\overset{def}{=} v - \nabla(-\Delta)^{-1}\theta,$$

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they overcome this difficulty and improved the regularity of u, and then obtained the gradient estimate of (u, v, θ). It is clear that the Laplacian term, $-\Delta u$, plays a key role, while the method does not appear to be able to extend to the case without $-\Delta u$ even if the initial data is small.

However, for (1.1) with small data, we can get the global well-posedness which is the main result of this paper. The details can be given as follows:

Theorem 1.1. *Let $\alpha > 0$ and $\eta > 0$. Consider (1.1) with initial data $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$, $s > 2$, and $\mathrm{div}u_0 = 0$. There exists a small constant $\epsilon = \epsilon(\alpha, \eta) > 0$ such that if*

$$\|u_0\|_{H^s(\mathbb{R}^2)} + \|v_0\|_{H^s(\mathbb{R}^2)} + \|\theta_0\|_{H^s(\mathbb{R}^2)} < \epsilon, \tag{1.2}$$

then (1.1) admits a unique global solution (u, v, θ) satisfying

$$(u, v, \theta) \in C([0, \infty); H^s(\mathbb{R}^2)), \ \nabla v \in L^2([0, \infty); H^s(\mathbb{R}^2)).$$

Remark 1.2. *(i) If we neglect the coupled terms $\nabla\theta$ and $\mathrm{div}v$, the key part of (1.1) is the first two equations, which are very similar to the 2D MHD equations. To the best of our knowledge for that without velocity dissipation and small data, the global regularity result is empty (see, e.g.,* [2],[3],[5] *and references therein), from which, it seems very difficult to drop the condition (1.2).*

(ii) Motivated by the recent works on the local well-posedness for the non-resistive MHD equations (i.e., only with $-\Delta u$) with low regular initial data (see, e.g., [4],[6] *and* [19]*), we expect that similar result holds for the version of (1.1) with $-\Delta u$, $\alpha = 0$ and $\eta = 0$.*

Now, let us explain the difficulty and our idea. By the standard energy method, we can show that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(u, v, \theta)\|^2_{H^s(\mathbb{R}^2)} + \alpha\|(u, v)\|^2_{H^s(\mathbb{R}^2)} + \frac{\eta}{2}\|\nabla v\|^2_{H^s(\mathbb{R}^2)} \\ &\quad \le C(\eta)\left\{\|(\nabla u, \nabla v, \nabla\theta)\|_{L^\infty(\mathbb{R}^2)} + \|v\|^2_{L^\infty(\mathbb{R}^2)}\right\}\|(u, v, \theta)\|^2_{\dot{H}^s(\mathbb{R}^2)}, \end{aligned} \tag{1.3}$$

from which, we can see that (1.3) does not be closed under small initial data unless some norm of θ such as $\|\theta\|_{H^s(\mathbb{R}^2)}$ occurs on the left hand side of (1.3).

Our proof is exploiting the structure of (1.1). To make our idea clear, we give the details for the key part of linearized (1.1):

$$\begin{cases} \partial_t v + \alpha v - \eta\Delta v - \nabla\theta = 0, \\ \partial_t\theta - \mathrm{div}v = 0. \end{cases} \tag{1.4}$$

Applying the operator $\Lambda^{-1}\mathrm{div}$ and $\eta\Lambda$ to the first and second equation of (1.4), respectively, then adding the resulting equations, denote

$$\mathcal{R} :\overset{def}{=} \Lambda^{-1}\mathrm{div}, \ \ \Omega :\overset{def}{=} \mathcal{R}v + \eta\Lambda\theta,$$

it is easy to deduce

$$\partial_t\Omega + \frac{1}{\eta}\Omega = (\frac{1}{\eta} - \alpha)\mathcal{R}v. \tag{1.5}$$

Multiplying the first equation of (1.4) by a large enough constant M, adding the resulting equation to (1.5) and combining with the L^2 bound of Riesz transform, we can find the hidden thermal diffusion and then overcome this difficulty.

Let us complete this section by describing the notations we shall use in this paper. **Notations** For A, B two operator, we denote $[A, B] = AB - BA$, the commutator between A and B. The uniform constant C, which may be different on different lines, is independent of the parameters such as α and η in (1.1), while the constant $C(\cdot)$ means a constant depends on the element(s) in bracket. In some places of this paper, we may use L^p, $\dot{H}^s$ (H^s) and $\dot{B}^s_{p,r}$ ($B^s_{p,r}$) to stand for $L^p(\mathbb{R}^d)$, $\dot{H}^s(\mathbb{R}^d)$ ($H^s(\mathbb{R}^d)$) and $\dot{B}^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}(\mathbb{R}^d)$), respectively. We shall denote by $(a|b)$ the L^2 inner product of a and b, and $(a|b)_{\dot{H}^s}$ stands for the standard $\dot{H}^s$ inner product of a and b, more precisely, $(a|b)_{\dot{H}^s} = (\Lambda^s a|\Lambda^s b)$.

2. Preliminaries

In this section, we give some necessary definitions and propositions.

The fractional Laplacian operator $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ ($\alpha \geq 0$) is defined through the Fourier transform, namely,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where the Fourier transform is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

Let $\mathfrak{B} = \{\xi \in \mathbb{R}^d,\ |\xi| \leq \frac{4}{3}\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^d,\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. Choose two nonnegative smooth radial function χ, φ supported, respectively, in $\mathfrak{B}$ and $\mathfrak{C}$ such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathfrak{F}^{-1}\varphi$ and $\tilde{h} = \mathfrak{F}^{-1}\chi$, where $\mathfrak{F}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) f(x-y)dy,$$

$$S_j f = \sum_{k\leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y) f(x-y)dy.$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to annulus $\{\xi :\ C_1 2^j \leq |\xi| \leq C_2 2^j\}$, and S_j is a frequency projection to the ball $\{\xi :\ |\xi| \leq C2^j\}$. One easily verifies that with our choice of φ

$$\Delta_j \Delta_k f = 0 \text{ if } |j-k| \geq 2 \ \text{ and } \ \Delta_j(S_{k-1} f \Delta_k f) = 0 \text{ if } |j-k| \geq 5.$$

Let us recall the definition of the Besov space.

Definition 2.1. *Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty]^2$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is defined by*

$$\dot{B}^s_{p,q}(\mathbb{R}^d) = \{f \in \mathfrak{S}'(\mathbb{R}^d);\ \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{sqj}\|\Delta_j f\|^q_{L^p(\mathbb{R}^d)})^{\frac{1}{q}}, & \text{for } \ 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j f\|_{L^p(\mathbb{R}^d)}, & \text{for } \ q = \infty, \end{cases}$$

and $\mathfrak{S}'(\mathbb{R}^d)$ denotes the dual space of $\mathfrak{S}(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d);\ \partial^\alpha \hat{f}(0) = 0;\ \forall\ \alpha \in\ \mathbb{N}^d$ multi-index$\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

Definition 2.2. *Let $s > 0$, and $(p,q) \in [1,\infty]^2$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ is defined by*

$$B^s_{p,q}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{B^s_{p,q}(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)}.$$

For the special case $p = q = 2$, we have

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \approx \|f\|_{\dot{B}^s_{2,2}(\mathbb{R}^d)},$$

where $a \approx b$ means $C^{-1}b \le a \le Cb$ for some positive constant C, and the $\dot{H}^s(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ $(s > 0)$ norm of f can be also defined as follows:

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} :\overset{def}{=} \|\Lambda^s f\|_{L^2(\mathbb{R}^d)}$$

and

$$\|f\|_{H^s(\mathbb{R}^d)} :\overset{def}{=} \|f\|_{L^2(\mathbb{R}^d)} + \|\Lambda^s f\|_{L^2(\mathbb{R}^d)}.$$

Lemma 2.3. *(i)*[13] *Let $s > 0$, $1 \le p, r \le \infty$, then*

$$\|fg\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \le C\left\{\|f\|_{L^{p_1}(\mathbb{R}^d)}\|g\|_{\dot{B}^s_{p_2,r}(\mathbb{R}^d)} + \|g\|_{L^{r_1}(\mathbb{R}^d)}\|g\|_{\dot{B}^s_{r_2,r}(\mathbb{R}^d)}\right\}, \tag{2.1}$$

where $1 \le p_1, r_1 \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.
(ii)[14] *Let $s > 0$, and $1 < p < \infty$, then*

$$\|[\Lambda^s, f]g\|_{L^p(\mathbb{R}^d)} \le C\left\{\|\nabla f\|_{L^{p_1}(\mathbb{R}^d)}\|\Lambda^{s-1}g\|_{L^{p_2}(\mathbb{R}^d)} + \|\Lambda^s f\|_{L^{p_3}(\mathbb{R}^d)}\|g\|_{L^{p_4}(\mathbb{R}^d)}\right\} \tag{2.2}$$

where $1 < p_2, p_3 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

The following proposition and lemma provide Bernstein type inequalities for fractional derivatives and standard commutator estimate.

Proposition 2.4. *Let $\gamma \ge 0$. Let $1 \le p \le q \le \infty$.*

1) *If f satisfies*

$$supp\,\widehat{f} \subset \{\xi \in \mathbb{R}^d:\ |\xi| \le \mathcal{K}2^j\},$$

for some integer j and a constant $\mathcal{K} > 0$, then

$$\|(-\Delta)^\gamma f\|_{L^q(\mathbb{R}^d)} \le C_1(\gamma, p, q)\, 2^{2\gamma j + jd(\frac{1}{p} - \frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d :\ \mathcal{K}_1 2^j \le |\xi| \le \mathcal{K}_2 2^j\}$$

for some integer j and constants $0 < \mathcal{K}_1 \le \mathcal{K}_2$, then

$$C_1(\gamma, p, q)\, 2^{2\gamma j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\gamma f\|_{L^q(\mathbb{R}^d)} \le C_2(\gamma, p, q)\, 2^{2\gamma j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

Lemma 2.5. [1] *Let θ be a C^1 function on $\mathbb{R}^d$ such that $(1 + |\cdot|)\widehat{\theta} \in L^1(\mathbb{R}^d)$. There exists a constant C such that for any Lipschitz function a with gradient in $L^p(\mathbb{R}^d)$ and any function b in $L^q(\mathbb{R}^d)$, we have for any positive λ,*

$$\|[\theta(\lambda^{-1}D), a]b\|_{L^r(\mathbb{R}^d)} \le C\lambda^{-1} \|\nabla a\|_{L^p(\mathbb{R}^d)} \|b\|_{L^q(\mathbb{R}^d)}, \quad \text{with } \frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{2.3}$$

For more details about Besov space and Sobolev space such as some useful embedding inequalities, we refer to [1], [9] and [18].

The rest of this section is devoted to the proof of a generalized commutator estimate in Besov space. Firstly, we need a lemma.

Lemma 2.6. *Let $1 \le p, p_1, p_2 \le \infty$ satisfying $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. If $xh \in L^{p_1}(\mathbb{R}^d)$, $\nabla f \in L^\infty(\mathbb{R}^d)$ and $g \in L^{p_2}(\mathbb{R}^d)$, then*

$$\|h \star (fg) - f(h \star g)\|_{L^p(\mathbb{R}^d)} \le C\|xh\|_{L^{p_1}(\mathbb{R}^d)} \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^{p_2}(\mathbb{R}^d)}, \tag{2.4}$$

where C is a constant independent of f, g, h.

Proof of Lemma 2.6. (2.4) can be proved by using the idea of Lemma 2.1 in [20], so we omit the details. □

Proposition 2.7. *Let $s \ge 0$, $\sigma > -1$ and $1 \le p, r \le \infty$, then*

$$\|[\Lambda^s, f \cdot \nabla]g\|_{\dot{B}^\sigma_{p,r}(\mathbb{R}^d)} \le C\left\{\|\nabla f\|_{L^\infty(\mathbb{R}^d)} \|g\|_{\dot{B}^{\sigma+s}_{p,r}(\mathbb{R}^d)} + \|\nabla g\|_{L^\infty(\mathbb{R}^d)} \|f\|_{\dot{B}^{\sigma+s}_{p,r}(\mathbb{R}^d)}\right\}, \tag{2.5}$$

where $\mathrm{div} f = 0$ *and the constant C is independent of f and g.*

For the proof, we shall use homogeneous Bony's decomposition:

$$uv = \sum_{j\in\mathbb{Z}} S_{j-1}u\Delta_j v + \sum_{j\in\mathbb{Z}} \Delta_j u S_{j-1} v + \sum_{j\in\mathbb{Z}} \Delta_j u \tilde{\Delta}_j v,$$

where $\tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, which is applied to split the commutator $\Theta = [\Delta_j, u]v$ as follows:

$$\begin{aligned}\Theta = &\sum_{|k-j|\le 4} [\Delta_j, S_{k-1}u]\Delta_k v + \sum_{|k-j|\le 4} \Delta_j(\Delta_k u\ S_{k-1}v)\\ &+ \sum_{k\ge j-2} \Delta_k u\ \Delta_j S_{k+2} v + \sum_{k\ge j-3} \Delta_j(\Delta_k u\ \tilde{\Delta}_k v).\end{aligned}$$

If we replace Λ^s by Riesz operator $\Lambda^{-1}\partial_1$ or the operator $\Lambda^{-\alpha}\partial_1$ $(0 < \alpha < 1)$ in (2.5), [10]-[12] established some similar estimates, which play the essential role in the proof of the global well-posedness for 2D Boussinesq equations.

Proof of Proposition 2.7. It suffices to prove the case $1 \le r < \infty$, the case $r = \infty$ can be bounded similarly. In this proof, $(c_j)_{j\in\mathbb{Z}}$ is a generic element of $l^r(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}} c_j^r \le 1$. **1** stands for the characteristic function. We split the left hand side of (2.5) into two terms.

$$\begin{aligned}
\|[\Lambda^s, f\cdot\nabla]g\|_{\dot B^{\sigma}_{p,r}} \le\, & C\left(\sum_{j\in\mathbb{Z}} 2^{j\sigma r}\|\Delta_j(f\cdot\nabla\Lambda^s g) - f\cdot\nabla\Delta_j\Lambda^s g\|_{L^p}^r\right)^{\frac{1}{r}}\\
& + C\left(\sum_{j\in\mathbb{Z}} 2^{j\sigma r}\|\Delta_j\Lambda^s(f\cdot\nabla g) - f\cdot\nabla\Delta_j\Lambda^s g\|_{L^p}^r\right)^{\frac{1}{r}}\\
\le & C\left(\sum_{j\in\mathbb{Z}} 2^{j\sigma r}\|[\Delta_j, f\cdot\nabla]\Lambda^s g\|_{L^p}^r\right)^{\frac{1}{r}} + C\left(\sum_{j\in\mathbb{Z}} 2^{j\sigma r}\|[\Delta_j\Lambda^s, f\cdot\nabla]\Lambda^s g\|_{L^p}^r\right)^{\frac{1}{r}}\\
= & K_1 + K_2.
\end{aligned} \tag{2.6}$$

The Estimate of K_1. Using the homogeneous Bony's decomposition,

$$\begin{aligned}
\|[\Delta_j, f\cdot\nabla]\Lambda^s g\|_{L^p} \le & \sum_{|k-j|\le 4}\|[\Delta_j, S_{k-1}f\cdot\nabla]\Delta_k\Lambda^s g\|_{L^p} + \sum_{|k-j|\le 4}\|\Delta_j(\Delta_k f\cdot\nabla S_{k-1}\Lambda^s g)\|_{L^p}\\
& + \sum_{k\ge j-2}\|\Delta_k f\cdot\nabla\Delta_j S_{k+2}\Lambda^s g\|_{L^p} + \sum_{k\ge j-3}\|\Delta_j(\Delta_k f\cdot\nabla\tilde\Delta_k\Lambda^s g)\|_{L^p}\\
= & K_{11} + K_{12} + K_{13} + K_{14}.
\end{aligned}$$

Thanks to (2.3) and Bernstein's inequality,

$$\begin{aligned}
K_{11} \le & C\sum_{|k-j|\le 4}\|\nabla S_{k-1}f\|_{L^\infty}\|\Delta_k\Lambda^s g\|_{L^p}\\
\le & C2^{-j\sigma}\|\nabla f\|_{L^\infty}\sum_{|k-j|\le 4} 2^{(j-k)\sigma}2^{k\sigma}\|\Delta_k\Lambda^s g\|_{L^p}\\
\le & C2^{-j\sigma}\|\nabla f\|_{L^\infty}\|\Lambda^s g\|_{\dot B^{\sigma}_{p,r}}\sum_{|k-j|\le 4} 2^{(j-k)\sigma}c_k\\
\le & Cc_j 2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot B^{\sigma+s}_{p,r}}.
\end{aligned}$$

By Hölder's inequality and Bernstein's inequality, we get for $s \ge 0$,

$$\begin{aligned}
K_{12} \le & C\sum_{|k-j|\le 4} 2^{ks}\|\nabla S_{k-1}g\|_{L^\infty}\|\Delta_k f\|_{L^p}\\
\le & C\|\nabla g\|_{L^\infty}\sum_{|k-j|\le 4} 2^{ks}\|\Delta_k f\|_{L^p}\\
\le & C2^{-j\sigma}\|\nabla g\|_{L^\infty}\sum_{|k-j|\le 4} 2^{(j-k)\sigma}2^{k(\sigma+s)}\|\Delta_k f\|_{L^p}\\
\le & C2^{-j\sigma}\|\nabla g\|_{L^\infty}\|f\|_{\dot B^{\sigma+s}_{p,r}}\sum_{|k-j|\le 4} 2^{(j-k)\sigma}c_k\\
= & Cc_j 2^{-j\sigma}\|\nabla g\|_{L^\infty}\|f\|_{\dot B^{\sigma+s}_{p,r}}.
\end{aligned}$$

For the term K_{13},

$$\begin{aligned}K_{13} \leq& C2^{j(1+s)}\|\Delta_j g\|_{L^p}\sum_{k\geq j-2}\|\Delta_k f\|_{L^\infty}\\ \leq& C2^{j(1+s)}\|\Delta_j g\|_{L^p}\sum_{k\geq j-2}2^{-k}\|\nabla\Delta_k f\|_{L^\infty}\\ \leq& C2^{js}\|\Delta_j g\|_{L^p}\|\nabla f\|_{L^\infty}\\ =& Cc_j 2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{s+\sigma}_{p,r}}.\end{aligned}$$

Using $\mathrm{div} f = 0$, Bernstein's inequality and Hölder's inequality,

$$\begin{aligned}K_{14} \leq& 2^j\sum_{k\geq j-3}\|\Delta_k f\|_{L^\infty}\|\Lambda^s\tilde{\Delta}_k g\|_{L^p}\\ \leq& C2^{-j\sigma}\sum_{k\geq j-3}2^{j(\sigma+1)-k}\|\nabla\Delta_k f\|_{L^\infty}\|\Lambda^s\tilde{\Delta}_k g\|_{L^p}\\ \leq& C2^{-j\sigma}\|\nabla f\|_{L^\infty}\sum_{k\geq j-3}2^{(j-k)(\sigma+1)}2^{k\sigma}\|\Lambda^s\tilde{\Delta}_k g\|_{L^p}\\ \leq& C2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma+s}_{p,r}}\sum_{k\geq j-3}2^{(j-k)(\sigma+1)}c_k\\ =& Cc_j 2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma+s}_{p,r}},\end{aligned}$$

where we have used Young's inequality for series for the last equality, namely, $\forall\sigma > -1$,

$$\sum_{k\in\mathbb{Z}}\sum_{k\geq j-3}2^{(j-k)(\sigma+1)}c_k \leq \|2^{-(\sigma+1)k}\mathbf{1}_{k\geq -3}\|_{l^1(\mathbb{Z})}\|c_k\|_{l^1(\mathbb{Z})} \leq C.$$

Thus,

$$K_1 \leq \sum_{1\leq i\leq 4}\Big(\sum_{j\in\mathbb{Z}}2^{j\sigma r}K_{1i}^r\Big)^{\frac{1}{r}} \leq C\left\{\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma+s}_{p,r}} + \|\nabla g\|_{L^\infty}\|f\|_{\dot{B}^{\sigma+s}_{p,r}}\right\}.$$

The Estimate of K_2. Using the homogeneous Bony's decomposition again,

$$\begin{aligned}\|[\Delta_j\Lambda^s, f\cdot\nabla]g\|_{L^p} \leq& \sum_{|k-j|\leq 4}\|[\Delta_j\Lambda^s, S_{k-1}f\cdot\nabla]\Delta_k g\|_{L^p} + \sum_{|k-j|\leq 4}\|\Delta_j\Lambda^s(\Delta_k f\cdot\nabla S_{k-1}g)\|_{L^p}\\ &+\sum_{k\geq j-2}\|\Delta_k f\cdot\nabla\Delta_j S_{k+2}\Lambda^s g\|_{L^p} + \sum_{k\geq j-3}\|\Delta_j\Lambda^s(\Delta_k f\cdot\nabla\tilde{\Delta}_k)g\|_{L^p}\\ =& K_{21}+K_{22}+K_{23}+K_{24}.\end{aligned}$$

Since

$$\widehat{\Delta_j\Lambda^s f}(\xi) = \varphi(2^{-j}\xi)|\xi|^s\widehat{f}(\xi) = 2^{js}\varphi(2^{-j}\xi)|2^{-j}\xi|^s\widehat{f}(\xi),$$

we can represent $\Delta_j\Lambda^s f$ as a convolution, namely,

$$\Delta_j\Lambda^s f(x) = \{2^{j(d+s)}\zeta(2^j\cdot)\star f\}(x),\ \text{for some } \zeta\in\mathcal{S}(\mathbb{R}^d).$$

(2.3) can not be used to bound K_{21}, but thanks to (2.4), and using Bernstein's inequality,

$$\begin{aligned}
K_{21} \leq & C\|x2^{j(d+s)}\zeta(2^jx)\|_{L^1}\sum_{|k-j|\leq 4}\|\nabla S_{k-1}f\|_{L^\infty}\|\nabla\Delta_k g\|_{L^p}\\
\leq & C2^{j(s-1)}\|\nabla f\|_{L^\infty}\sum_{|k-j|\leq 4}\|\nabla\Delta_k g\|_{L^p}\\
\leq & C2^{-j\sigma}\|\nabla f\|_{L^\infty}\sum_{|k-j|\leq 4}2^{(j-k)(s+\sigma-1)}2^{k(s+\sigma-1)}\|\nabla\Delta_k g\|_{L^p}\\
= & C2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma+s}_{p,r}}\sum_{|k-j|\leq 4}2^{(j-k)(s+\sigma-1)}c_k\\
= & Cc_j2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma+s}_{p,r}}.
\end{aligned}$$

For the terms K_{22} and K_{23}, with a similar procedure as the estimate of K_{12} and K_{13}, respectively, we have

$$\begin{aligned}
K_{22} \leq & C2^{js}\sum_{|k-j|\leq 4}\|\nabla S_{k-1}g\|_{L^\infty}\|\Delta_k f\|_{L^p}\\
\leq & C2^{js}\|\nabla g\|_{L^\infty}\sum_{|k-j|\leq 4}\|\Delta_k f\|_{L^p}\\
\leq & C2^{-j\sigma}\|\nabla g\|_{L^\infty}\sum_{|k-j|\leq 4}2^{(j-k)(s+\sigma)}2^{k(s+\sigma)}\|\Delta_k f\|_{L^p}\\
\leq & C2^{-j\sigma}\|\nabla g\|_{L^\infty}\|f\|_{\dot{B}^{\sigma+s}_{p,r}}\sum_{|k-j|\leq 4}2^{(j-k)(s+\sigma)}c_k\\
= & Cc_j2^{-j\sigma}\|\nabla g\|_{L^\infty}\|f\|_{\dot{B}^{\sigma+s}_{p,r}},
\end{aligned}$$

and

$$\begin{aligned}
K_{23} \leq & C2^{j(1+s)}\|\Delta_j g\|_{L^p}\sum_{k\geq j-2}\|\Delta_k f\|_{L^\infty}\\
\leq & C2^{j(1+s)}\|\Delta_j g\|_{L^p}\sum_{k\geq j-2}2^{-k}\|\nabla\Delta_k f\|_{L^\infty}\\
\leq & C2^{js}\|\Delta_j g\|_{L^p}\|\nabla f\|_{L^\infty}\\
\leq & C2^{-j\sigma}2^{j(s+\sigma)}\|\Delta_j g\|_{L^p}\|\nabla f\|_{L^\infty}\\
= & Cc_j2^{-j\sigma}\|\nabla f\|_{L^\infty}\|g\|_{\dot{B}^{\sigma+s}_{p,r}}.
\end{aligned}$$

Using $\mathrm{div} f = 0$, Bernstein's inequality, Hölder's inequality and Young's inequality for series,

$$\begin{aligned}
K_{24} \le & C2^{j(s+1)} \sum_{k\ge j-3} \|\Delta_k f\|_{L^\infty} \|\tilde{\Delta}_k g\|_{L^p} \\
\le & C2^{j(s+1)} \sum_{k\ge j-3} 2^{-k} \|\nabla \Delta_k f\|_{L^\infty} \|\tilde{\Delta}_k g\|_{L^p} \\
\le & C2^{j(s+1)} \|\nabla f\|_{L^\infty} \sum_{k\ge j-3} 2^{-k} \|\tilde{\Delta}_k g\|_{L^p} \\
\le & C2^{-j\sigma} \|\nabla f\|_{L^\infty} \sum_{k\ge j-3} 2^{(j-k)(s+1+\sigma)} 2^{k(s+\sigma)} \|\tilde{\Delta}_k g\|_{L^p} \\
= & C2^{-j\sigma} \|\nabla f\|_{L^\infty} \|g\|_{\dot{B}^{\sigma+s}_{p,r}} \sum_{k\ge j-3} 2^{(j-k)(s+1+\sigma)} c_k \quad (s+\sigma > -1) \\
= & Cc_j 2^{-j\sigma} \|\nabla f\|_{L^\infty} \|g\|_{\dot{B}^{\sigma+s}_{p,r}}.
\end{aligned}$$

It is easy to deduce that

$$K_2 \le \sum_{1\le i\le 4} (\sum_{j\in\mathbb{Z}} 2^{j\sigma r} K_{2i}^r)^{\frac{1}{r}} \le C \left\{ \|\nabla f\|_{L^\infty} \|g\|_{\dot{B}^{\sigma+s}_{p,r}} + \|\nabla g\|_{L^\infty} \|f\|_{\dot{B}^{\sigma+s}_{p,r}} \right\}.$$

Combining with the estimates of K_1 and K_2 leads the desired estimate (2.5). □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by splitting the details into two steps. In step 1, we show the local well-posedness for (1.1) in brief. More precisely, we only give local a priori bound since other details can be proved by standard method, see Chapter 3 [16]. In step 2, we find the hidden thermal diffusion by exploiting the structure as we described in section 1 and then obtain the global bound with small data by using the commutator estimate (2.5) in section 2.

Now, we begin the proof.

Step 1. Local a priori bound. Thanks to the cancelation property,

$$(\nabla\theta|v) + (\mathrm{div} v|\theta) = 0,$$

it is easy to get the L^2 bound of (u, v, θ):

$$\frac{1}{2}\frac{d}{dt}\|(u, v, \theta)\|_{L^2}^2 + \alpha\|(u, v)\|_{L^2}^2 + \eta\|\nabla v\|_{L^2}^2 = 0. \tag{3.1}$$

By the standard energy estimate, and noting

$$(\nabla\theta|v)_{\dot{H}^s} + (\mathrm{div} v|\theta)_{\dot{H}^s} = 0,$$

with (3.1), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u,v,\theta)\|_{H^s}^2+\alpha\|(u,v)\|_{H^s}^2+\eta\|\nabla v\|_{H^s}^2\\
&\quad=-(u\cdot\nabla u|u)_{\dot H^s}-(\mathrm{div}(v\otimes v)|u)_{\dot H^s}-(u\cdot\nabla v|v)_{\dot H^s}\\
&\qquad-(v\cdot\nabla u|v)-(u\cdot\nabla\theta|\theta)_{\dot H^s}\\
&\quad\stackrel{def}{:=}\sum_{i=1}^{5}I_i.
\end{aligned}\tag{3.2}$$

By integrating by parts, we get

$$(u\cdot\nabla\Lambda^s u|\Lambda^s u)=(u\cdot\nabla\Lambda^s v|\Lambda^s v)=(u\cdot\nabla\Lambda^s\theta|\Lambda^s\theta)=0$$

and

$$(v\cdot\nabla\Lambda^s u|\Lambda^s v)+(v\cdot\nabla\Lambda^s v|\Lambda^s u)=-(\mathrm{div}v|\Lambda^s u\cdot\Lambda^s v),$$

from which, with the equality $\mathrm{div}(v\otimes v)=v\ \mathrm{div}v+v\cdot\nabla v$, we obtain

$$\begin{aligned}
I_1=&([\Lambda^s,u\cdot\nabla]u|\Lambda^s u),\quad I_3=([\Lambda^s,u\cdot\nabla]v|\Lambda^s v),\quad I_5=([\Lambda^s,u\cdot\nabla]\theta|\Lambda^s\theta),\\
I_2+I_4=&(\mathrm{div}v|v\cdot u)_{\dot H^s}+(v\cdot\nabla v|u)_{\dot H^s}+(v\cdot\nabla u|v)_{\dot H^s}\\
=&(v\ \mathrm{div}v|u)_{\dot H^s}+([\Lambda^s,v\cdot\nabla]v|\Lambda^s u)+([\Lambda^s,v\cdot\nabla]u|\Lambda^s u)-(\mathrm{div}v|\Lambda^s u\cdot\Lambda^s v).
\end{aligned}$$

Using Hölder's inequality, (2.1), (2.2) and Young's inequality follows that

$$\begin{aligned}
I_1\le&C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2,\\
I_2+I_4\le&C(\|\mathrm{div}v\|_{L^\infty}\|v\|_{\dot H^s}+\|v\|_{L^\infty}\|\mathrm{div}v\|_{\dot H^s})\|u\|_{\dot H^s}\\
&+C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|u\|_{\dot H^s}^2+\|v\|_{\dot H^s}^2)\\
&+C\|\mathrm{div}v\|_{L^\infty}\|u\|_{\dot H^s}\|v\|_{\dot H^s}\\
\le&C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|u\|_{\dot H^s}^2+\|v\|_{\dot H^s}^2)\\
&+C\eta\|v\|_{L^\infty}^2\|u\|_{\dot H^s}^2+\frac{\eta}{2}\|\nabla v\|_{\dot H^s}^2,\\
I_3\le&C(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty})(\|u\|_{\dot H^s}^2+\|v\|_{\dot H^s}^2),\\
I_5\le&C(\|\nabla u\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})(\|u\|_{\dot H^s}^2+\|\theta\|_{\dot H^s}^2).
\end{aligned}$$

Combining with the above estimates in (3.2), thanks to

$$\|f\|_{\dot H^s}\le C\|\nabla f\|_{H^{s-1}},\quad \|\nabla f\|_{L^\infty}\le C\|\nabla f\|_{H^{s-1}},\quad s>2,$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u,v,\theta)\|_{H^s}^2+\alpha\|(u,v)\|_{H^s}^2+\frac{\eta}{2}\|\nabla v\|_{H^s}^2\\
&\le C(\|\nabla u\|_{H^{s-1}}^2+\|\nabla v\|_{H^{s-1}}^2+\|\nabla\theta\|_{H^{s-1}}^2)^{\frac{3}{2}}+C\eta\|v\|_{H^s}^2\|u\|_{H^s}^2\\
&\le C(\eta)\left(\|(u,v,\theta)\|_{H^s}^3+\|(u,v)\|_{H^s}^4\right),
\end{aligned}\tag{3.3}$$

which implies that there exists a $T_0=T_0(\eta,\|(u_0,v_0,\theta_0)\|_{H^s})>0$ such that $\forall\ t\in(0,T_0]$,

$$\|u(t)\|_{H^s}^2+\|v(t)\|_{H^s}^2+\|\theta(t)\|_{H^s}^2\le C(\eta,\ T_0,\|(u_0,v_0,\theta_0)\|_{H^s})$$

and then

$$\alpha\int_0^t\|(u,v)(\tau)\|_{H^s}^2d\tau+\frac{\eta}{2}\int_0^t\|\nabla v(\tau)\|_{H^s}^2d\tau\le C(\eta,\ T_0,\|(u_0,v_0,\theta_0)\|_{H^s}).$$

So we can get the local a priori bound and then obtain the local well-posedness of (1.1) by standard method.

Step 2. Global well-posedness. Thanks to step 1, it suffices to give the global a priori bound. Denote

$$\mathcal{R} :\stackrel{def}{=} \Lambda^{-1}\mathrm{div}, \quad \Omega :\stackrel{def}{=} \mathcal{R}v + \eta\Lambda\theta,$$

and we will use the L^2 bound of Riesz transform $\mathcal{R}$ in some places of this step.
Applying the operator $\mathcal{R}$ and Λ to the second equation and third equation of (1.1), respectively, we get

$$\partial_t \mathcal{R}v + u\cdot\nabla\mathcal{R}v + \alpha\mathcal{R}v + \eta\Lambda\mathrm{div}v + \Lambda\theta = -[\mathcal{R}, u\cdot\nabla]v - \mathcal{R}(v\cdot\nabla u) \tag{3.4}$$

$$\partial_t\Lambda\theta + u\cdot\nabla\Lambda\theta - \Lambda\mathrm{div}v = -[\Lambda, u\cdot\nabla]\theta. \tag{3.5}$$

Multiplying (3.5) by η and adding the resulting equation to (3.4) lead

$$\partial_t\Omega + u\cdot\nabla\Omega + \frac{1}{\eta}\Omega = (\frac{1}{\eta} - \alpha)\mathcal{R}v - [\mathcal{R}, u\cdot\nabla]v - \mathcal{R}(v\cdot\nabla u) - \eta[\Lambda, u\cdot\nabla]\theta. \tag{3.6}$$

Multiplying (3.6) by Ω, integrating in $\mathbb{R}^2$ and using

$$(u\cdot\nabla\Omega|\Omega) = 0$$

follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2 + \frac{1}{\eta}\|\Omega\|_{L^2}^2 = &(\frac{1}{\eta} - \alpha)(\mathcal{R}v|\Omega)\\ &- ([\mathcal{R}, u\cdot\nabla]v|\Omega) - (\mathcal{R}(v\cdot\nabla u)|\Omega) - \eta([\Lambda, u\cdot\nabla]\theta|\Omega)\\ :\stackrel{def}{=} &J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{3.7}$$

By Hölder's inequality and Young's inequality,

$$\begin{aligned}|J_1| \le&|\frac{1}{\eta} - \alpha|\|v\|_{L^2}\|\Omega\|_{L^2} \le C\eta|\frac{1}{\eta} - \alpha|^2\|v\|_{L^2}^2 + \frac{1}{8\eta}\|\Omega\|_{L^2}^2,\\ |J_2| \le&\|[\mathcal{R}, u\cdot\nabla]v\|_{L^2}\|\Omega\|_{L^2} \le C\|u\|_{L^\infty}\|\nabla v\|_{L^2}\|\Omega\|_{L^2}\\ \le&C\eta\|u\|_{L^\infty}^2\|\nabla v\|_{L^2}^2 + \frac{1}{8\eta}\|\Omega\|_{L^2}^2,\\ |J_3| \le&C\|\mathcal{R}(v\cdot\nabla u)\|_{L^2}\|\Omega\|_{L^2} \le C\eta\|v\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \frac{1}{8\eta}\|\Omega\|_{L^2}^2.\end{aligned}$$

By (2.2), using $\eta\Lambda\theta = \Omega - \mathcal{R}v$ and Young's inequality,

$$\begin{aligned}|J_4| \le&C\eta(\|\nabla u\|_{L^\infty}\|\Lambda\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda u\|_{L^2})\|\Omega\|_{L^2}\\ \le&C\|\nabla u\|_{L^\infty}(\|\Omega\|_{L^2} + \|\mathcal{R}v\|_{L^2})\|\Omega\|_{L^2} + C\eta\|\nabla\theta\|_{L^\infty}\|\Lambda u\|_{L^2}\|\Omega\|_{L^2}\\ \le&C\eta\|\nabla u\|_{L^\infty}^2(\|v\|_{L^2}^2 + \|\Omega\|_{L^2}^2) + C\eta^3\|\nabla\theta\|_{L^\infty}^2\|\Lambda u\|_{L^2}^2 + \frac{1}{8\eta}\|\Omega\|_{L^2}^2.\end{aligned}$$

Inserting the above estimates into (3.7) and absorbing the four $\frac{1}{8\eta}\|\Omega\|_{L^2}^2$ by the left hand side of the resulting inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Omega\|_{L^2}^2+\frac{1}{2\eta}\|\Omega\|_{L^2}^2 &\le C\eta|\frac{1}{\eta}-\alpha|^2\|v\|_{L^2}^2\\ &\quad+C(\eta+\eta^3)(\|(u,\nabla u,v)\|_{L^\infty}^2+\|\nabla\theta\|_{L^\infty}^2)(\|(u,v)\|_{H^1}^2+\|\Omega\|_{L^2}^2)\\ &\le C\eta|\frac{1}{\eta}-\alpha|^2\|v\|_{L^2}^2+C(\eta+\eta^3)\|(u,v,\theta)\|_{H^s}^2(\|(u,v)\|_{H^1}^2+\|\Omega\|_{L^2}^2).\end{aligned}\tag{3.8}$$

Next, we give the $\dot{H}^{s-1}$ bound of Ω. Applying Λ^{s-1} to the both sides of (3.6), and taking the inner product with $\Lambda^{s-1}\Omega$, while thanks to

$$(u\cdot\nabla\Lambda^{s-1}\Omega|\Lambda^{s-1}\Omega)=0,$$

we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Omega\|_{\dot{H}^{s-1}}^2+\frac{1}{\eta}\|\Omega\|_{\dot{H}^{s-1}}^2 &=(\frac{1}{\eta}-\alpha)(\mathcal{R}v|\Omega)_{\dot{H}^{s-1}}-([\Lambda^{s-1},u\cdot\nabla]\Omega|\Lambda^{s-1}\Omega)\\ &\quad-([\mathcal{R},u\cdot\nabla]v|\Omega)_{\dot{H}^{s-1}}-(\mathcal{R}(v\cdot\nabla u)|\Omega)_{\dot{H}^{s-1}}-\eta([\Lambda,u\cdot\nabla]\theta|\Omega)_{\dot{H}^{s-1}}\\ &\overset{def}{:=}L_1+L_2+L_3+L_4+L_5.\end{aligned}\tag{3.9}$$

By Hölder's inequality and Young's inequality,

$$|L_1|\le C|\frac{1}{\eta}-\alpha|\|\Lambda^{s-1}v\|_{L^2}\|\Omega\|_{\dot{H}^{s-1}}\le C\eta|\frac{1}{\eta}-\alpha|^2\|v\|_{\dot{H}^{s-1}}^2+\frac{1}{8\eta}\|\Omega\|_{\dot{H}^{s-1}}^2.$$

By Hölder's inequality, (2.2), Young's inequality and using $\eta\Lambda\theta=\Omega-\mathcal{R}v$,

$$\begin{aligned}|L_2|\le&\|[\Lambda^{s-1},u\cdot\nabla]\theta\|_{L^2}\|\Lambda^{s-1}\Omega\|_{L^2}\\ \le&C(\|\nabla u\|_{L^\infty}\|\Lambda^{s-1}\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\Lambda^{s-1}u\|_{L^2})\|\Lambda^{s-1}\Omega\|_{L^2}\\ \le&C\eta(\|\nabla u\|_{L^\infty}^2\|\Lambda^{s-1}\theta\|_{L^2}^2+\|\nabla\theta\|_{L^\infty}^2\|\Lambda^{s-1}u\|_{L^2}^2)+\frac{1}{8\eta}\|\Omega\|_{\dot{H}^{s-1}}^2\\ \le&C\eta(\|\nabla u\|_{L^\infty}^2\|\Lambda\theta\|_{\dot{H}^{s-2}}^2+\|\nabla\theta\|_{L^\infty}^2\|\Lambda^{s-1}u\|_{L^2}^2)+\frac{1}{8\eta}\|\Omega\|_{\dot{H}^{s-1}}^2\\ \le&\frac{C}{\eta}\|\nabla u\|_{L^\infty}^2(\|\Omega\|_{\dot{H}^{s-2}}^2+\|v\|_{\dot{H}^{s-2}}^2)+C\eta\|\nabla\theta\|_{L^\infty}^2\|\Lambda^{s-1}u\|_{L^2}^2+\frac{1}{8\eta}\|\Omega\|_{\dot{H}^{s-1}}^2.\end{aligned}$$

By Hölder's inequality, (2.1) and Young's inequality, $\forall\ \iota\in(0,s-2)$,

$$\begin{aligned}|L_3|\le&\|\mathcal{R}(u\cdot\nabla v)-u\cdot\nabla\mathcal{R}v\|_{\dot{H}^{s-1}}\|\Omega\|_{\dot{H}^{s-1}}\\ \le&C\left\{\|u\|_{L^\infty}\|v\|_{\dot{H}^s}+(\|\nabla v\|_{L^\infty}+\|\nabla\mathcal{R}v\|_{L^\infty})\|u\|_{\dot{H}^{s-1}}\right\}\|\Omega\|_{\dot{H}^{s-1}}\\ \le&C(\|u\|_{L^\infty}\|v\|_{\dot{H}^s}+\|\nabla v\|_{H^{1+\iota}}\|u\|_{\dot{H}^{s-1}})\|\Omega\|_{\dot{H}^{s-1}}\ (\iota\in(0,s-2))\\ \le&C\eta(\|u\|_{L^\infty}^2\|v\|_{\dot{H}^s}^2+\|\nabla v\|_{H^{1+\iota}}^2\|u\|_{\dot{H}^{s-1}}^2)+\frac{1}{8\eta}\|\Omega\|_{\dot{H}^{s-1}}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}|L_4|\le&C\|v\cdot\nabla u\|_{\dot{H}^{s-1}}\|\Omega\|_{\dot{H}^{s-1}}\\ \le&C(\|v\|_{L^\infty}\|\nabla u\|_{\dot{H}^{s-1}}+\|\nabla u\|_{L^\infty}\|v\|_{\dot{H}^{s-1}})\|\Omega\|_{\dot{H}^{s-1}}\\ \le&C\eta(\|v\|_{L^\infty}^2\|u\|_{\dot{H}^s}^2+\|\nabla u\|_{L^\infty}\|v\|_{\dot{H}^{s-1}}^2)+\frac{1}{8\eta}\|\Omega\|_{\dot{H}^{s-1}}^2.\end{aligned}$$

By Hölder's inequality and thanks to the commutator estimates (2.5) with $d=2$, $s=1$, $\sigma=s-1$, $p=r=2$,

$$\begin{aligned}|L_5| \le&\eta(\|\nabla\theta\|_{L^\infty}\|u\|_{\dot H^s}+\|\nabla u\|_{L^\infty}\|\theta\|_{\dot H^s})\|\Omega\|_{\dot H^s}\\ \le&C\eta^3(\|\nabla\theta\|^2_{L^\infty}\|u\|^2_{\dot H^s}+\|\nabla u\|^2_{L^\infty}\|\theta\|^2_{\dot H^s})+\frac{1}{8\eta}\|\Omega\|^2_{\dot H^s}\\ \le&C\eta^3\left\{\|\nabla\theta\|^2_{L^\infty}\|u\|^2_{\dot H^s}+\frac{1}{\eta^2}\|\nabla u\|^2_{L^\infty}(\|\Omega\|^2_{\dot H^{s-1}}+\|v\|^2_{\dot H^{s-1}})\right\}+\frac{1}{8\eta}\|\Omega\|^2_{\dot H^s}\\ \le&C\eta^3\|\nabla\theta\|^2_{L^\infty}\|u\|^2_{\dot H^s}+C\eta\|\nabla u\|^2_{L^\infty}(\|\Omega\|^2_{\dot H^{s-1}}+\|v\|^2_{\dot H^{s-1}})+\frac{1}{8\eta}\|\Omega\|^2_{\dot H^s}.\end{aligned}$$

Combining the estimates of L_l $(l=1,2,3,4,5)$ in (3.9), it is easy to deduce that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\Omega\|^2_{\dot H^{s-1}}+\frac{1}{2\eta}\|\Omega\|^2_{\dot H^{s-1}}\le C\eta|\frac1\eta-\alpha|^2\|v\|^2_{\dot H^{s-1}}+C(\frac1\eta+\eta+\eta^3)\\ &\quad\times\left(\|(u,v,\nabla u,\nabla v)\|^2_{L^\infty}+\|\nabla\theta\|^2_{L^\infty}+\|\nabla v\|^2_{H^{1+\iota}}\right)\left(\|(u,v)\|^2_{H^s}+\|\Omega\|^2_{\dot H^{s-1}}\right),\end{aligned}$$

and thanks to (3.8), we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|\Omega\|^2_{H^{s-1}}+\frac{1}{2\eta}\|\Omega\|^2_{H^{s-1}}\le C\eta|\frac1\eta-\alpha|^2\|v\|^2_{H^{s-1}}+C(\frac1\eta+\eta+\eta^3)\\ \times\|(u,v,\theta)\|^2_{H^s}\left(\|(u,v)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}\right).\end{aligned}\tag{3.10}$$

Using $\eta\Lambda\theta=\Omega-\mathcal{R}v$ again, we can rewrite (3.3) as follows:

$$\begin{aligned}&\frac12\frac{d}{dt}\|(u,v,\theta)\|^2_{H^s}+\alpha\|(u,v)\|^2_{H^s}+\frac\eta2\|\nabla v\|^2_{H^s}\le C\|(\nabla u,\nabla v,\nabla\theta)\|_{H^{s-1}}\\ &\quad\times\left(\|\nabla u\|^2_{H^{s-1}}+\|\nabla v\|^2_{H^{s-1}}+\|\Lambda\theta\|^2_{H^{s-1}}\right)+C\eta\|v\|^2_{H^s}\|u\|^2_{H^s}\\ &\le C(1+\frac{1}{\eta^2})\|(u,v,\theta)\|_{H^s}\|(\nabla u,\nabla v,v,\Omega)\|^2_{H^{s-1}}+C\eta\|v\|^2_{H^s}\|u\|^2_{H^s}.\end{aligned}\tag{3.11}$$

Multiplying (3.11) by

$$M\overset{def}{:=}\frac{2C\eta|\frac1\eta-\alpha|^2}{\alpha},$$

adding the resulting inequality to (3.10), and then absorbing the term $C\eta|\frac1\eta-\alpha|^2\|v\|^2_{H^{s-1}}$, we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\left\{M\|(u,v,\theta)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}\right\}+\frac{M\alpha}{2}\|(u,v)\|^2_{H^s}+\frac{M\eta}{2}\|\nabla v\|^2_{H^s}+\frac{1}{2\eta}\|\Omega\|^2_{H^{s-1}}\\ &\le C(M,\eta)\left(\|(u,v,\theta)\|^2_{H^s}+\|(u,v,\theta)\|_{H^s}\right)\left(\|(u,v)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}\right).\end{aligned}$$

Denote

$$m\overset{def}{:=}\min\left\{\frac{M\alpha}{2},\ \frac{\eta M}{2},\ \frac{1}{2\eta}\right\},$$

by Young's inequality, it is easy to get

$$\begin{aligned}&\frac{d}{dt}\left\{M\|(u,v,\theta)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}\right\}+2m(\|(u,\nabla v,v)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}})\\ &\le C(M,\eta)\left(\|(u,v,\theta)\|^2_{H^s}+\|(u,v,\theta)\|_{H^s}\right)\left(\|(u,v)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}\right)\\ &\le C(M,m,\eta)\|(u,v,\theta)\|^2_{H^s}\left(\|(u,v)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}\right)+m(\|(u,v)\|^2_{H^s}+\|\Omega\|^2_{H^{s-1}}).\end{aligned}\tag{3.12}$$

Absorbing the second term on the right hand side in the last inequality of (3.12), then integrating in time yields $\forall\ t>0$,

$$\begin{aligned}M\|(u,v,\theta)(t)\|_{H^s}^2+\|\Omega(t)\|_{H^{s-1}}^2+m\int_0^t\|(u,\nabla v,v)(\tau)\|_{H^s}^2+\|\Omega(\tau)\|_{H^{s-1}}^2d\tau\\ \le C(M,m,\eta)\int_0^t\|(u,v,\theta)(\tau)\|_{H^s}^2\left(\|(u,v)(\tau)\|_{H^s}^2+\|\Omega(\tau)\|_{H^{s-1}}^2\right)d\tau\\ +M\|(u_0,v_0,\theta_0)\|_{H^s}^2+\|\Omega_0\|_{H^{s-1}}^2,\end{aligned} \tag{3.13}$$

With small data (1.2), choosing ϵ to be so small that

$$M\|(u_0,v_0,\theta_0)\|_{H^s}^2+\|\Omega_0\|_{H^{s-1}}^2\le\frac{Mm}{2C(M,m,\eta)}$$

which implies that $\|(u_0,v_0,\theta_0)\|_{H^s}^2<\frac{m}{2C(M,m,\eta)}$. Suppose there exists a first time $T^\star$ such that $\forall t\in(0,T^\star)$,

$$M\|(u,v,\theta)(t)\|_{H^s}^2+\|\Omega(t)\|_{H^{s-1}}^2\le\frac{Mm}{2C(M,m,\eta)}$$

and

$$\lim_{\epsilon_1\searrow 0}M\|(u,v,\theta)(T^\star-\epsilon_1)\|_{H^s}^2+\|\Omega(T^\star-\epsilon_1)\|_{H^{s-1}}^2>\frac{Mm}{2C(M,m,\eta)}. \tag{3.14}$$

However, from (3.13), we can deduce

$$\begin{aligned}M\|(u,v,\theta)(T^\star-\epsilon_1)\|_{H^s}^2+\|\Omega(T^\star-\epsilon_1)\|_{H^{s-1}}^2\\ +\frac{m}{2}\int_0^{T^\star-\epsilon_1}\|(u,\nabla v,v)(\tau)\|_{H^s}^2+\|\Omega(\tau)\|_{H^{s-1}}^2d\tau\\ \le M\|(u_0,v_0,\theta_0)\|_{H^s}^2+\|\Omega_0\|_{H^{s-1}}^2<\frac{Mm}{2C(M,m,\eta)},\end{aligned} \tag{3.15}$$

which yields that

$$M\|(u,v,\theta)(T^\star-\epsilon_1)\|_{H^s}^2+\|\Omega(T^\star-\epsilon_1)\|_{H^{s-1}}^2<\frac{Mm}{2C(M,m,\eta)},$$

from which, and taking $\epsilon_1\searrow 0$, we get a contradiction with (3.14). Therefore,

$$\lim_{\epsilon_1\searrow 0}M\|(u,v,\theta)(T^\star-\epsilon_1)\|_{H^s}^2+\|\Omega(T^\star-\epsilon_1)\|_{H^{s-1}}^2\le\frac{Mm}{2C(M,m,\eta)},$$

which indicates under condition (1.2), we have a global solution (u,v,θ) satisfying $\forall\ t>0$,

$$\|(u,v,\theta)(t)\|_{H^s}\le\frac{m}{2C(M,m,\eta)}=C(\alpha,\eta)<\infty,$$

and then using (3.13), we can also obtain

$$\int_0^t\|(u,\nabla v,v)(\tau)\|_{H^s}^2d\tau\le\frac{M}{2C(M,m,\eta)}=C(\alpha,\eta)<\infty.$$

This completes the proof of Theorem 1.1.

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